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VARIOUS CENTROIDS OF QUADRILATERALS WITHOUT SYMMETRY

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Abstract. For a quadrilateral P, we consider the centroid G_0 of the vertices of P, the perimeter centroid G_1 of the edges of P and the centroid G_2 of the interior of P, respectively. It is well known that P satisfies $G_0 = G_1$ or $G_0 = G_2$ if and only if it is a parallelogram. In this paper, we investigate various quadrilaterals satisfying $G_1 = G_2$. As a result, we establish some characterization theorems. One of them asserts the existence of convex quadrilaterals satisfying $G_1 = G_2$ without symmetry.

1. Introduction

Suppose that P denotes a quadrilateral. We consider the centroid G_0 of the vertices of P, the centroid G_1 of the edges of P and the centroid G_2 of the interior of P, respectively. The centroid G_1 of the edges of P is also called the perimeter centroid of P ([3, 4]). Then, in ([10]) the following characterization theorem was given:

Proposition 1.1. Let P denote a quadrilateral. Then the following are equivalent.

(1) P satisfies $G_0 = G_1$.

(2) P satisfies $G_0 = G_2$.

(3) P is a parallelogram.

Obviously, every parallelogram satisfies $G_0 = G_1 = G_2 (= M)$, where M denotes the intersection point of diagonals.

Hence, it is quite natural to ask the following ([10]):

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Question 1.2. Which quadrilaterals satisfy $G_1 = G_2$?

In this regard, we have the following characterization ([8]).

Proposition 1.3. Suppose that P denotes a convex quadrilateral whose two diagonals are perpendicular to each other. We denote by M the intersection point of diagonals of P. Then we have the following.

(1) P satisfies $G_1 = G_2(= M)$ if and only if P is a rhombus.

(2) If P satisfies $G_1 = G_2(\neq M)$, then P is a kite.

A kite is a quadrilateral whose four sides can be grouped into two pairs of equal-length sides that are adjacent to each other. A kite, as defined above, may be either convex or concave, but the word *kite* is often restricted to the convex variety. A concave kite is called an *arrowhead*. A convex quadrilateral is a kite if and only if one diagonal is the perpendicular bisector of the other diagonal. In [8], kites satisfying $G_1 = G_2$ were completely classified. See also Proposition 1.4 below.

Furthermore, recently in [14] following the study in [1, 8, 10], the following characterizations were established. For graphs, see Figure 1 and 2.

Proposition 1.4. Let us denote by P a kite or an arrowhead. Then P satisfies $G_1 = G_2$ if and only if it is one of the following:

(1) P is a rhombus.

(2) P is similar to the following quadrilateral $ABCD$ defined by

$$A(1, 0), B(s, t), C(-1, 0), D(s, -t),$$

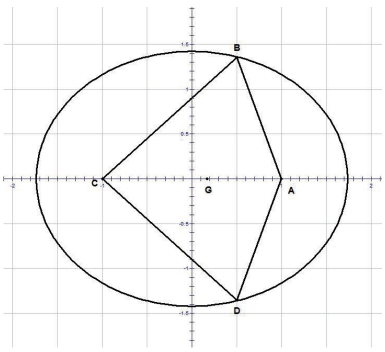
where $B(s, t)$ is a point on the ellipse $E : x^2/3 + y^2/2 = 1$ with $s \neq 0$ and $t > 0$.

Proposition 1.5. A trapezoid $P = ABCD$ with $AD \parallel BC$ satisfies $G_1 = G_2$ if and only if it is one of the following:

(1) P is a parallelogram.

(2) P is an isosceles trapezoid with $\overline{AB} = \overline{CD} = \overline{AD} + \overline{BC}$.

Proposition 1.6. Let us denote by P a circumscribed quadrilateral. If P satisfies $G_1 = G_2$, then it is a kite.



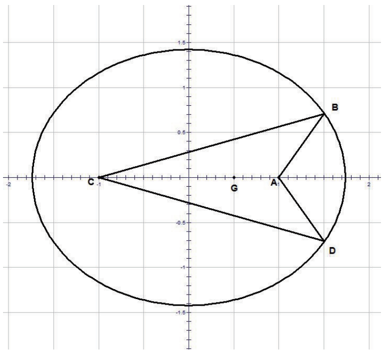


FIGURE 1. A kite($s = 1/2$) and an arrowhead ($s = 3/2$) satisfying $G_1 = G_2 (= G)$.

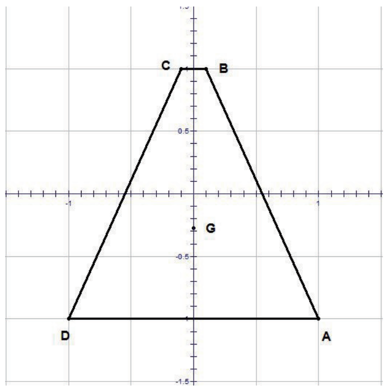


FIGURE 2. An isosceles trapezoid satisfying $G_1 = G_2 (= G)$.

Proposition 1.7. Suppose that a convex quadrilateral P has two pairs of adjacent edges such that the total length of a pair equals to that of the other pair. If P satisfies $G_1 = G_2$, then it is one of the following:

(1) P is a parallelogram.

(2) P is a kite.

Note that every quadrilateral satisfying $G_1 = G_2$ given in the above Propositions has a symmetry.

Thus, it is also quite natural to ask the following:

Question 1.8. Does there exist a convex quadrilateral without symmetry satisfying $G_1 = G_2$?

In this paper, we investigate various quadrilaterals satisfying $G_1 = G_2$. First of all, in Section 2 we prove the following characterization theorem.

Proposition A. Suppose that a convex quadrilateral P has a pair of opposite edges of equal length. If P satisfies $G_1 = G_2$, then it is one of the following:

(1) P is a parallelogram.

(2) P is an isosceles trapezoid.

For an isosceles trapezoid satisfying $G_1 = G_2$, see Proposition 1.5.

Next, surprisingly enough in Section 3 we answer Question 1.8 affirmatively. In fact, we prove the following characterization theorem which asserts the existence of convex quadrilaterals satisfying $G_1 = G_2$ without symmetry.

Theorem B. Suppose that a convex quadrilateral P has a pair of adjacent edges of equal length. If P satisfies $G_1 = G_2$, then it is one of the following:

(1) P is a kite.

(2) P is similar to the following quadrilateral $ABCD$ given by

$$A(x, y), B(0, a), C(-1, 0), D(0, -a),$$

where for a constant $a \in (1/\sqrt{3}, 1/\sqrt{2})$, x and y are defined by

(1.1)
$$x = \frac{1}{a^2}\{1 - a^2 - \sqrt{(1-2a^2)(1+a^2)}\}, \quad 2y^2 = (2a^2 - x)(2x + 1).$$

Conversely, such quadrilaterals are convex ones satisfying $G_1 = G_2$ without symmetry.

For a kite satisfying $G_1 = G_2$, see Proposition 1.4.

Finally, in Section 4 we investigate concave quadrilaterals with a pair of adjacent edges of equal length which satisfy $G_1 = G_2$. In contrast to the convex case in Theorem B, we prove the following characterization theorem.

Theorem C. Suppose that a concave quadrilateral P has a pair of adjacent edges of equal length. If P satisfies $G_1 = G_2$, then it is an arrowhead.

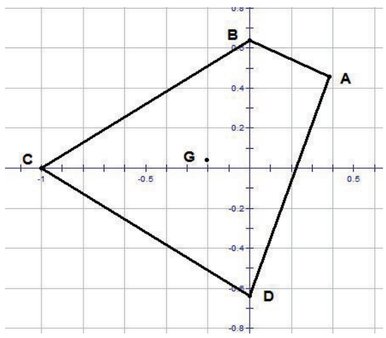


FIGURE 3. A convex quadrilateral without symmetry satisfying $G_1 = G_2(= G)$.

For an arrowhead satisfying $G_1 = G_2$, see Proposition 1.4.

In [11], it was proved that among quadrilaterals parallelograms are the only ones satisfying $G_1 = M$, where M is the intersection point of diagonals.

Suppose that P is a triangle. Then the centroid G_1 coincides with the center of the Spieker circle, which is the incircle of the triangle formed by connecting midpoint of each side of the original triangle P ([3, p. 249]). In this case, the centroid G_0 always coincides with the centroid $G_2(= G)$, where $G = (A + B + C)/3$. Furthermore, the perimeter centroid G_1 of P satisfies $G_1 = G_2$ if and only if the triangle P is equilateral ([13, Theorem 2]).

Now suppose that P is a polygon. Then the geometric method to find the centroid G_2 of P was given in [5]. In [12], mathematical definitions of centroid G_2 of planar bounded domains were given. For higher dimensions, it was shown that the centroid G_0 of the vertices of a simplex in an n-dimensional space always coincides with the centroid G_n of the interior of the simplex ([2, 13]).

Archimedes established some area properties of parabolic sections and then formulated the centroid of parabolic sections ([15]). Using these properties, some characterizations of parabolas were given in [6, 7, 9].

2. Preliminaries and Proposition A

In this section, first of all we recall the various centroids of a quadrilateral. For various centroids of a quadrilateral $ABCD$, we have the following.

Lemma 2.1. Let us denote by P the (convex or concave) quadrilateral $ABCD$. Then we have the following.

(1) The centroid G_0 of P is given by

$$G_0 = \frac{A+B+C+D}{4}. \tag{2.1}$$

(2) The centroid G_1 of P is given by

$$G_1 = \frac{(l_4+l_1)A+(l_1+l_2)B+(l_2+l_3)C+(l_3+l_4)D}{2l}, \tag{2.2}$$

where we put $l_1 = AB, l_2 = BC, l_3 = CD, l_4 = DA$ and $l = l_1+l_2+l_3+l_4$.

(3) If the area of the quadrilateral $ABCD$ is given by $m = \beta \pm \delta$, where $\beta = \triangle BCD$ and $\delta = \triangle ABD$, then the centroid G_2 of P is given by

$$G_2 = \frac{\pm\delta A + mB + \beta C + mD}{3m}. \tag{2.3}$$

Proof. It is straightforward to prove (1), (2) and (3) or see [5, 10].

Now, we prove Proposition A stated in Section 1.

Suppose that a convex quadrilateral P has a pair of opposite edges of equal length. That is, the quadrilateral $P = ABCD$ satisfies $BC = AD$. Using a similarity transformation if necessary, we may introduce a coordinates system so that the vertices of P are given by

$$A(x,y), B(0,a), C(-1,0), D(0,-b), \tag{2.4}$$

where a, b and x are positive real numbers.

It follows from Lemma 2.1 that the perimeter centroid G_1 of P is given by

$$G_1 = \frac{1}{2l}(x(l_1+l_4)-(l_2+l_3), y(l_1+l_4)+a(l_1+l_2)-b(l_3+l_4)), \tag{2.5}$$

where we put by l the perimeter of P with

(2.6)
$$l_1 = \sqrt{x^2 + (y-a)^2}, l_2 = \sqrt{a^2+1}, l_3 = \sqrt{b^2+1}, l_4 = \sqrt{x^2 + (y+b)^2}.$$

The centroid G_2 of P is given by

$$G_2 = \frac{1}{3(x+1)}(x^2 - 1, xy + (a-b)(x+1)). \tag{2.7}$$

Since the quadrilateral P satisfies $BC = AD$, we have

$$l_4 = l_2. \tag{2.8}$$

Now, $G_1 = G_2$ is equivalent to the following:

$$l_1 + l_4 = \frac{2x+1}{x+2}(l_2 + l_3), \tag{2.9}$$

$$al_1 - bl_4 = \frac{-y + (a-2b)x - 2b}{x+2} l_2 + \frac{-y + (2a-b)x + 2a}{x+2} l_3. \tag{2.10}$$

It follows from (2.8) and (2.9) that

$$l_1 = \frac{x-1}{x+2} l_2 + \frac{2x+1}{x+2} l_3. \tag{2.11}$$

Let us substitute l_1 and l_4 in (2.11) and (2.8) resp., into (2.10). Then, we get

$$(y + bx - a)(l_2 + l_3) = 0. \tag{2.12}$$

It follows from (2.12) that $y = -bx + a$, and hence the edges AB and CD are parallel to each other. Hence, Proposition 1.5 implies that the quadrilateral P is either a parallelogram or an isosceles trapezoid. This completes the proof of Proposition A.

3. Convex quadrilaterals with a pair of adjacent edges of equal length

In this section, we prove Theorem B stated in Section 1.

Suppose that $P = ABCD$ denotes a convex quadrilateral with a pair of adjacent edges of equal length. Then, we may assume that $BC = CD$. Using a similarity transformation if necessary, we may introduce a coordinates system so that the vertices of P are given by

$$A(x, y), B(0, a), C(-1, 0), D(0, -a), \tag{3.1}$$

where a and x are positive real numbers. Using the symmetry with respect to the x-axis, we may assume that $y \geq 0$. In case $y = 0$, the quadrilateral is a kite. Hence, hereafter we assume that $y > 0$.

It follows from Lemma 2.1 that the perimeter centroid G_1 of P is given by

$$G_1 = \frac{1}{2l}(xl_1 + xl_4 - l_2 - l_3, yl_1 + yl_4 + al_1 - al_4), \tag{3.2}$$

where we put by l the perimeter of P with

$$l_1 = \sqrt{x^2 + (y-a)^2}, l_2 = l_3 = \sqrt{a^2+1}, l_4 = \sqrt{x^2 + (y+a)^2}. \tag{3.3}$$

The centroid G_2 of P is given by

$$G_2 = \frac{1}{3(x+1)}(x^2 - 1, xy). \tag{3.4}$$

Suppose that $G_1 = G_2$. Then we have

$$l_1 + l_4 = 2\sqrt{a^2+1}\frac{2x+1}{x+2} \tag{3.5}$$

and

$$-l_1 + l_4 = 2\sqrt{a^2+1}\frac{y}{a(x+2)}. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$l_1 = \sqrt{a^2+1}\frac{a(2x+1) - y}{a(x+2)}, \tag{3.7}$$

and

$$l_4 = \sqrt{a^2+1}\frac{a(2x+1) + y}{a(x+2)}. \tag{3.8}$$

Squaring both of (3.7) and (3.8) and using the definitions of l_1 and l_4 in (3.3) resp., we obtain

(3.9)

$$y^2(a^2x^2 + 4a^2x + 3a^2 - 1)$$
$$- 2ay\{a^2x^2 + 2(a^2-1)x + 3a^2 - 1\} + a^2(x^2-1)(x^2 + 4x - 3a^2 + 1) = 0.$$

and

(3.10)

$$y^2(a^2x^2 + 4a^2x + 3a^2 - 1)$$
$$+ 2ay\{a^2x^2 + 2(a^2-1)x + 3a^2 - 1\} + a^2(x^2-1)(x^2 + 4x - 3a^2 + 1) = 0.$$

It follows from (3.9) and (3.10) that

$$4ay\{a^2x^2 + 2(a^2-1)x + 3a^2 - 1\} = 0. \tag{3.11}$$

Since a and y are positive, (3.11) implies that x is a positive root of the following quadratic polynomial

$$a^2x^2 + 2(a^2 - 1)x + 3a^2 - 1 = 0. \tag{3.12}$$

Hence, from $D/4 = (1 - 2a^2)(a^2 + 1)$ we have

$$a^2 \leq \frac{1}{2} \tag{3.13}$$

and x is given by

$$x = \frac{1}{a^2}\left(1 - a^2 \pm \sqrt{1 - a^2 - 2a^4}\right). \tag{3.14}$$

Together with (3.12), (3.10) shows

$$y^2(a^2x^2 + 4a^2x + 3a^2 - 1) = a^2(1 - x^2)(x^2 + 4x - 3a^2 + 1). \tag{3.15}$$

It follows from (3.12) and (3.15) that

$$2y^2 = a^2(1 - x^2)(x + 2). \tag{3.16}$$

With the help of (3.12), (3.16) can be rewritten as

$$2y^2 = (2a^2 - x)(2x + 1). \tag{3.17}$$

Now, we consider the following two cases.

Case 1.

$$x = \frac{1}{a^2}\left(1 - a^2 + \sqrt{1 - a^2 - 2a^4}\right). \tag{3.18}$$

Then we have

$$2a^2 - x = \frac{1}{a^2}\{f(a^2) - \sqrt{-f(a^2)}\}, \tag{3.19}$$

where we put $f(s) = 2s^2 + s - 1$. Since $a^2 \leq 1/2$, we get $f(a^2) \leq 0$. This shows that $2a^2 - x \leq 0$, which contradicts (3.17) because $y > 0$. Thus, this case can not occur.

Case 2.

$$x = \frac{1}{a^2}\left(1 - a^2 - \sqrt{1 - a^2 - 2a^4}\right). \tag{3.20}$$

Since $x > 0$, it follows from (3.20) that $1/3 < a^2$. Hence, we get from (3.13) that

$$\frac{1}{3} < a^2 \leq \frac{1}{2}. \tag{3.21}$$

If $a^2 = 1/2$, then (3.20) shows that $x = 1$. Hence, it follows from (3.16) that $y = 0$, which leads a contradiction. Thus, we have

$$\frac{1}{3} < a^2 < \frac{1}{2}. \tag{3.22}$$

In this case, (3.20) implies

$$2a^2 - x = \frac{1}{a^2}\{f(a^2) + \sqrt{-f(a^2)}\}, \tag{3.23}$$

where $f(s) = 2s^2 + s - 1$. If we put $t = f(s) = 2s^2 + s - 1$, then we have

$$\frac{1}{3} < s < \frac{1}{2} \Leftrightarrow -\frac{4}{9} < t < 0. \tag{3.24}$$

Note that the function $g(t) = t + \sqrt{-t}$ satisfies $g(t) > 0$ for all $t \in (-4/9, 0)$. Hence, for every positive number a satisfying (3.22), there exist positive numbers x and y satisfying (3.20) and (3.17), respectively.

Next, we show that for every positive number a satisfying (3.22), the quadrilateral is convex. Note that the quadrilateral is convex if and only if

$$\begin{aligned} &\text{slope of } \overleftrightarrow{AB} < \text{slope of } \overleftrightarrow{BC} \\ &\Leftrightarrow \frac{y-a}{x} < a \Leftrightarrow y < a(x+1). \end{aligned} \tag{3.25}$$

Using (3.17), it is straightforward to show that

$$y < a(x+1). \tag{3.26}$$

This implies that the quadrilateral is convex.

Conversely, we show that for every positive number a satisfying (3.22), the convex quadrilateral satisfies $G_1 = G_2$. For every positive number a satisfying (3.22), the positive numbers x and y defined by (3.20) and (3.17), respectively satisfies (3.12), and hence it is straightforward to show that x and y satisfy (3.9) and (3.10). Note that the right hand side of (3.8) is positive. It follows from (3.26) that the right hand side of (3.7) is also positive. Hence, x and y satisfies (3.7) and (3.8), which implies that the convex quadrilateral satisfies $G_1 = G_2$.

Furthermore, it is straightforward to show that opposite edges of the convex quadrilateral defined above are not parallel to each other, respectively. Thus, we see that the convex quadrilateral has no symmetries. This completes the proof of Theorem B.

Example 3.1. We put $a^2 = 0.4$. Then, it follows from (3.17) and (3.20) that

$$a \doteq 0.64, x \doteq 0.38, y \doteq 0.46. \tag{3.28}$$

Hence, we get

$$A \doteq (0.38, 0.46), B \doteq (0, 0.64), C \doteq (-1, 0), D \doteq (0, -0.64) \tag{3.29}$$

and

$$G_1 = G_2 = G \doteq (-0.21, 0.04). \tag{3.30}$$

See Figure 3 in Section 1.

4. Concave quadrilaterals with a pair of adjacent edges of equal length

In this section, we prove Theorem C stated in Section 1.

For concave quadrilaterals, in a similar argument as in the proof of Theorem B in Section 3, we proceed to prove Theorem C as follows.

Suppose that $P = ABCD$ denotes a concave quadrilateral with a pair of adjacent edges of equal length. Then, we may assume that $BC = CD$. Using a similarity transformation if necessary, we may introduce a coordinates system so that the vertices of P are given by

$$A(x, y), B(0, a), C(-1, 0), D(0, -a), \tag{4.1}$$

where a is a positive real number. Using the symmetry with respect to the x-axis, we may assume that $y \geq 0$. In case $y = 0$, the quadrilateral is an arrowhead. Hence, hereafter we suppose that $y > 0$.

If $P = ABCD$ is a concave quadrilateral with $x > 0$, that is, the vertex B lies in the interior of the triangle ACD, then the proof of Theorem B shows that $P = ABCD$ can not satisfy $G_1 = G_2$. Henceforth, we assume that $x < 0$.

It follows from Lemma 2.1 that the perimeter centroid G_1 of P is given by

$$G_1 = \frac{1}{2l}(xl_1 + xl_4 - l_2 - l_3, yl_1 + yl_4 + al_1 - al_4), \tag{4.2}$$

where we put by l the perimeter of P with

$$l_1 = \sqrt{x^2 + (y-a)^2}, l_2 = l_3 = \sqrt{a^2 + 1}, l_4 = \sqrt{x^2 + (y+a)^2}. \tag{4.3}$$

Even if $x < 0$, it follows from Lemma 2.1 that the centroid G_2 of the concave quadrilateral P is also given by

$$G_2 = \frac{1}{3(x+1)}(x^2-1, xy). \tag{4.4}$$

Suppose that $G_1 = G_2$. Then we have

$$l_1 + l_4 = 2\sqrt{a^2+1}\frac{2x+1}{x+2} \tag{4.5}$$

and

$$-l_1 + l_4 = 2\sqrt{a^2+1}\frac{y}{a(x+2)}. \tag{4.6}$$

Combining (4.5) and (4.6), we get

$$l_1 = \sqrt{a^2+1}\frac{a(2x+1)-y}{a(x+2)}, \tag{4.7}$$

and

$$l_4 = \sqrt{a^2+1}\frac{a(2x+1)+y}{a(x+2)}. \tag{4.8}$$

Squaring both of (4.7) and (4.8) and using the definitions of l_1 and l_4 in (4.3) resp., we obtain

$$\begin{aligned} &y^2(a^2x^2+4a^2x+3a^2-1)\\ &-2ay\{a^2x^2+2(a^2-1)x+3a^2-1\}+a^2(x^2-1)(x^2+4x-3a^2+1)=0. \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &y^2(a^2x^2+4a^2x+3a^2-1)\\ &+2ay\{a^2x^2+2(a^2-1)x+3a^2-1\}+a^2(x^2-1)(x^2+4x-3a^2+1)=0. \end{aligned} \tag{4.10}$$

It follows from (4.9) and (4.10) that

$$4ay\{a^2x^2+2(a^2-1)x+3a^2-1\}=0. \tag{4.11}$$

Since a and y are positive, (4.11) shows that x is a negative root of the following quadratic polynomial

$$a^2x^2+2(a^2-1)x+3a^2-1=0. \tag{4.12}$$

Hence, from $D/4=(1-2a^2)(a^2+1)\geq 0$ we have

$$a^2 \leq \frac{1}{2} \tag{4.13}$$

and x is given by

$$x = \frac{1}{a^2}\left(1 - a^2 \pm \sqrt{(1-2a^2)(1+a^2)}\right). \tag{4.14}$$

Together with (4.12), (4.10) shows

$$y^2(a^2x^2 + 4a^2x + 3a^2 - 1) = a^2(1-x^2)(x^2 + 4x - 3a^2 + 1). \tag{4.15}$$

It follows from (4.12) and (4.15) that

$$2y^2 = a^2(1-x^2)(x+2). \tag{4.16}$$

With the help of (4.12), (4.16) can be rewritten as

$$2y^2 = (2a^2 - x)(2x+1). \tag{4.17}$$

Now, we consider the following two cases.

Case 1.

$$x = \frac{1}{a^2}\left(1 - a^2 + \sqrt{(1-2a^2)(1+a^2)}\right). \tag{4.18}$$

Then we have

$$2a^2 - x = \frac{1}{a^2}\{(2a^2-1)(1+a^2) - \sqrt{(1-2a^2)(1+a^2)}\}. \tag{4.19}$$

Since $a^2 \leq 1/2$, from (4.19) we get $2a^2 - x \leq 0$. This is a contradiction because $x < 0$. Thus, this case can not occur.

Case 2.

$$x = \frac{1}{a^2}\left(1 - a^2 - \sqrt{(1-2a^2)(1+a^2)}\right). \tag{4.20}$$

Since $x < 0$, it follows from (4.20) that

$$0 < a^2 < \frac{1}{3}. \tag{4.21}$$

Furthermore, (4.17) shows that

$$-\frac{1}{2} < x < 0. \tag{4.22}$$

For every positive number a satisfying (4.21), there exist a negative number x and a positive number y satisfying (4.20) and (4.17), respectively. Hence the point $A(x, y)$ is well defined.

Finally, for a fixed positive number a satisfying (4.21) we show that $ABCD$ can not form a quadrilateral as follows. Since $y > 0$, it follows from (4.22) that $ABCD$ is a concave quadrilateral if and only if

$$\begin{aligned} &\text{slope of} \quad \overleftrightarrow{BC} < \text{slope of} \quad \overleftrightarrow{AB} \\ &\Leftrightarrow \quad a < \frac{y-a}{x} \quad \Leftrightarrow \quad y < a(x+1). \end{aligned} \tag{4.23}$$

Suppose that $ABCD$ is a concave quadrilateral. Then (4.23) shows that

$$2y^2 < 2a^2(x+1)^2. \tag{4.24}$$

It follows from (4.17) that (4.24) is equivalent to

$$2(a^2+1)x + 1 < 0. \tag{4.25}$$

Together with (4.20), this shows that for $s = a^2 \in (0, 1/3)$

$$g(s) > h(s) > 0, \tag{4.26}$$

where we put

$$g(s) = 2(s+1)\sqrt{(1-2s)(1+s)}, \quad h(s) = 2 + s - 2s^2. \tag{4.27}$$

On the other hand, a simple computation shows

$$g(s)^2 - h(s)^2 = -s^2(12s^2 + 16s + 5) < 0, \tag{4.28}$$

which contradicts (4.26). This contradiction implies that $ABCD$ cannot be a quadrilateral. Thus we must have $y = 0$, and hence the quadrilateral is an arrowhead. This completes the proof of Theorem C.

Remark 4.1. We put $a = 0.5$. Then, it follows from (4.20) and (4.17) that

$$x \doteq -0.46, y \doteq 0.69. \tag{4.29}$$

Hence, we get

$$A \doteq (-0.46, 0.69), B \doteq (0, 0.5), C \doteq (-1, 0), D \doteq (0, -0.5). \tag{4.30}$$

See Figure 4.

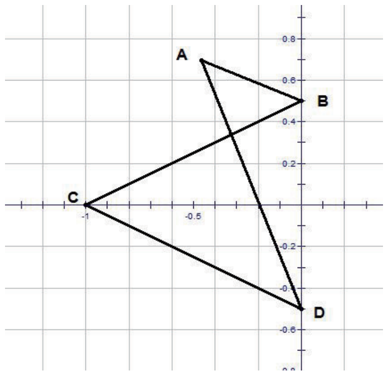


FIGURE 4. $ABCD$ is not a quadrilateral.

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